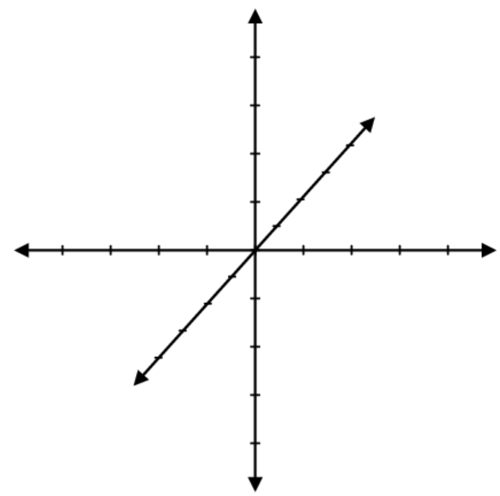
SECTION 17.6: PARAMETRIC SURFACES AND SURFACE AREA

In the same way we used vector-valued functions of **one variable**, $\vec{r}(t)$ to trace out **curves**, we can use vector-valued functions of **two variables** $\vec{r}(u, v)$ to trace out **surfaces** in space.

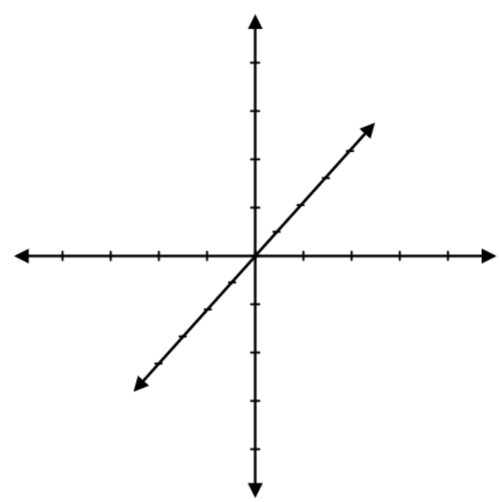
EXAMPLE 1: Convert the following parametric surfaces to rectangular coordinates to sketch the graph.

1. $\vec{r}(u, v) = \langle u, v, 6 - 2u - v \rangle$ for $0 \leq u \leq 3$ and $0 \leq v \leq 6 - 2u$.



Ans: The portion of $z = 6 - 2x - y$ which lies in the first octant.

2. $\vec{r}(u, v) = \left\langle u\cos(v), u\sin(v), u^2 \right\rangle$ for $u \geq 0$ and $0 \leq v < 2\pi$.



Ans: $z = x^2 + y^2$

FINDING PARAMETRIC DESCRIPTIONS

FUNCTIONS: If one variable is a function of the other two, we can use a 'cheap' parametrization:

- If $z = f(x, y)$, then let $x = u$ and $y = v$ so $z = f(u, v)$: $\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$
- If $y = g(x, z)$, then let $x = u$ and $z = v$ so $y = g(u, v)$: $\vec{r}(u, v) = \langle u, g(u, v), v\rangle$
- If $x = h(y, z)$, then let $y = u$ and $z = v$ so $x = h(u, v)$: $\vec{r}(u, v) = \langle h(u, v), u, v\rangle$

EXAMPLE 2: Find a parametric description for the graph of $y = 3x^2z - z^3$.

Ans: $\vec{r}(u, v) = \left\langle u, 3u^2v - v^3, v\right\rangle$

CYLINDRICAL COORDINATES: Recall the conversion to cylindrical coordinates:

$x = r\cos(\theta)$ $\qquad$ $y = r\sin(\theta)$ $\qquad$ $z = z$

EXAMPLE 3: Use cylindrical coordinates to help you find parametric representations for the following:

1. $x^2 + y^2 = 4$

   Ans: $\vec{r}(u, v) = \langle 2\cos(u), 2\sin(u), v\rangle$

2. $z = \sqrt{3x^2 + 3y^2}$.

   Ans: $\vec{r}(u, v) = \left\langle v\cos(u), v\sin(u), v\sqrt{3}\right\rangle$

SPHERICAL COORDINATES: Recall the conversion to spherical coordinates:

$$x = \rho \sin(\phi)\cos(\theta) \qquad y = \rho \sin(\phi)\sin(\theta) \qquad z = \rho\cos(\phi)$$

EXAMPLE 4: Use spherical coordinates to help you find parametric representations for the following:

1. $x^2 + y^2 + z^2 = 4$

Ans: $\vec{r}(u, v) = \langle 2\sin(u)\cos(v), 2\sin(u)\sin(v), 2\cos(u)\rangle$

2. $z = \sqrt{3x^2 + 3y^2}$.

Ans: $\vec{r}(u, v) = \left\langle \frac{1}{2}\, u\cos(v), \frac{1}{2}\, u\sin(v), \frac{\sqrt{3}}{2}\, u \right\rangle$

SURFACES OF REVOLUTION: Recall that if we revolve the graph of $y = f(x)$ about the x-axis, the resulting surface is called a **surface of revolution**. The surface may be envisioned as a series of circles parallel to the yz-plane, centered on the x-axis, with radius $f(x)$. Hence we may parametrize this surface using $x = u$ and thinking of a polar coordinates for y and z: $y = f(u)\cos(v)$, $z = f(u)\sin(v)$.

$$\vec{r}(u, v) = \langle u, f(u)\cos(v), f(u)\sin(v)\rangle$$

EXAMPLE 5: Find a parametric representation of the surface obtained by revolving $y = x^2$ about the x-axis.

Ans: $\vec{r}(u, v) = \left\langle u, u^2\cos(v), u^2\sin(v)\right\rangle$

EXAMPLE 6: Find a parametric representation of the surface obtained by revolving $x = \sin(y)$ about the y-axis.

Ans: $\vec{r}(u, v) = \langle \sin(u)\cos(v), u, \sin(u)\sin(v)\rangle$

DIFFERENTIAL CALCULUS OF PARAMETRIC SURFACES

QUESTION: Suppose $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$. How would you calculate $\vec{r}_u(u,v) = \dfrac{\partial}{\partial u}\vec{r}(u,v)$?

EXAMPLE 7: Find and simplify $\vec{r}_u(u,v)$, $\vec{r}_v(u,v)$, and $\vec{r}_u(u,v) \times \vec{r}_v(u,v)$ for $\vec{r}(u,v) = \left\langle u\cos(v), u\sin(v), u^2\right\rangle$.

1. $\vec{r}_u(u,v)$

Ans: $\vec{r}_u(u,v) = \langle \cos(v), \sin(v), 2u\rangle$

2. $\vec{r}_v(u,v)$

Ans: $\vec{r}_v(u,v) = \langle -u\sin(v), u\cos(v), 0\rangle$

3. $\vec{r}_u(u,v) \times \vec{r}_v(u,v)$

Ans: $\vec{r}_u(u,v) \times \vec{r}_v(u,v) = \left\langle -2u^2\cos(v), -2u^2\sin(v), u\right\rangle$

NORMAL VECTORS

THEOREM: If $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ then $(\vec{r}_u \times \vec{r}_v)(u_0, v_0)$ is normal to the surface $\vec{r}(u, v)$ at $\vec{r}(u_0, v_0)$.

EXAMPLE 8: Consider the surface S traced out by: $\vec{r}(u, v) = \left\langle u\cos(v), u\sin(v), u^2 \right\rangle$ for $u \geq 0$ and $0 \leq v < 2\pi$.

1. Find (u_0, v_0) which corresponds to the point $(x, y, z) = (0, 2, 4)$.

Ans: $(u, v) = \left(2, \dfrac{\pi}{2}\right)$

2. Find a normal vector to S at the point $(0, 2, 4)$.

Ans: $(\vec{r}_u \times \vec{r}_v)\left(2, \dfrac{\pi}{2}\right) = \langle 0, -8, 2 \rangle$

3. Find the equation of the tangent plane to S at $(0, 2, 4)$.

Ans: $8y - 2z = 8$

EXAMPLE 9: If $z = f(x, y)$ is parametrized as $\vec{r}(u, v) = \langle u, v, f(u, v) \rangle$, find $\vec{r}_u \times \vec{r}_v$. Does this look familiar?

Ans: $(\vec{r}_u \times \vec{r}_v)(u, v) = \langle -f_u(u, v), -f_v(u, v), 1 \rangle$

NOTE: In Chapter 15, we saw that the graph of $z = f(x, y)$ is the level surface of $F(x, y, z) = z - f(x, y) = 0$.

Hence, a normal vector can be found using the gradient: $\nabla F(x, y, z) = \langle -f_x(x, y), -f_y(x, y), 1 \rangle$.

SURFACE AREA

RECALL: A parametrization $\vec{r}$ of a curve C is **smooth** of $\vec{r}\,'(t)$ is continuous and $\|\vec{r}\,'(t)\| \neq 0$.

Smoothness of the parametrization guarantees $\hat{T}(t)$ exists (hence C has a consistent orientation.)

Moreover, if $\vec{r}$ is smooth, then the arc length differential $ds = \|\vec{r}\,'(t)\|\, dt$ and

$$\text{the length of } C = \int_C ds = \int_C \|\vec{r}\,'(t)\|\, dt$$

DEFINITION: A parametrization $\vec{r}$ of a surface S is **smooth** if $\vec{r}_u \times \vec{r}_v$ is continuous and $\|(\vec{r}_u \times \vec{r}_v)(u, v)\| \neq 0$.

Smoothness of the parametrization guarantees that a unit normal vector $\hat{N}(u, v)$ exists (which **orients** S.)

THEOREM: If a surface S is described by a smooth parametrization $\vec{r}(u, v)$ as (u, v) runs through a region R:

$$\text{Surface Area of } S = \iint_R \|(\vec{r}_u \times \vec{r}_v)(u, v)\|\, dA$$

In particular, if $z = f(x, y)$ has continuous first partials throughout a region R:

$$\text{Surface Area of } f(x, y) = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA$$

EXAMPLE 10: Find the area of the following surfaces.

1. $\vec{r}(u, v) = \left\langle u\cos(v), u\sin(v), u^2 \right\rangle$ for $1 \leq u \leq 2$ and $0 \leq v < \dfrac{\pi}{2}$

Ans: $\displaystyle\int_0^{\pi/2} \int_1^2 u\sqrt{4u^2 + 1}\, du\, dv = \frac{\pi\left(17\sqrt{17} - 5\sqrt{5}\right)}{24}$ units^2

2. The portion of $x + 2y + 3z = 6$ which lies in the first octant.

Ans: $\displaystyle\int_0^3 \int_0^{-2y+6} \frac{\sqrt{14}}{3}\, dx\, dy = 3\sqrt{14}$ units^2

MATH 2700: SURFACE INTEGRALS

RECALL: We generalized integrals over an interval, $\int_a^b f(x)\,dx$, to integrals over a curve: $\int_C f(x,y,z)\,ds$.

We now generalize integrals over a region in the plane, $\iint_R f(x,y)\,dA$, to integrals over a surface: $\iint_S f(x,y,z)\,dS$.

RECALL: Formulas for Surface Area differentials, dS:

- If $z = f(x,y)$: $dS = \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\,dA$
- If $y = g(x,z)$: $dS = \sqrt{1 + g_x(x,z)^2 + g_z(x,z)^2}\,dA$
- If $x = h(y,z)$: $dS = \sqrt{1 + h_y(y,z)^2 + h_z(y,z)^2}\,dA$
- If $\vec{r}(u,v)$: $dS = \|r_u(u,v) \times r_v(u,v)\|\,dA$

GOAL: Rewrite $\iint_S f(x,y,z)\,dS$ as a double integral in terms of just two variables.

EXAMPLE 11: Let S be the portion of $x + 2y + 3z = 6$ which lies in the first octant. Find $\iint_S x\,dS$

Ans: $\iint_S x\,dS = \int_0^3 \int_0^{-2y+6} \frac{x\sqrt{14}}{3}\,dx\,dy = 6\sqrt{14}$

EXAMPLE 12: Let S be the portion of $z = 4 - x^2 - y^2$ which lies in the first octant.

Set-up, but do not evaluate, a double integral in polar coordinates equivalent to $\displaystyle\iint_S y\,z\,dS$.

$$\text{Ans: } \int_0^{\pi/2}\int_0^2 (r\sin(\theta))(4-r^2)\sqrt{1+4r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 r^2\sin(\theta)(4-r^2)\sqrt{1+4r^2}\,dr\,d\theta$$

EXAMPLE 13: Let S be described by: $\vec{r}(u,v) = \langle 4\cos(u), 4\sin(u), v\rangle$, for $-\pi \le u \le \pi$ and $0 \le v \le 5$.

Find the average value of $f(x,y,z) = y^2$ on S.

$$\text{Ans: } \overline{f} = \frac{1}{40\pi}\int_{-\pi}^{\pi}\int_0^5 16\sin^2(u)\,4\,dv\,du = \frac{8}{5\pi}\int_{-\pi}^{\pi}\int_0^5 \sin^2(u)\,dv\,du = 8$$

FLUX INTEGRALS

RECALL: Circulation or work done **along** a curve C is given by $\int_C \vec{F} \cdot \hat{T}\, ds$. Flux **across** C is given by $\int_C \vec{F} \cdot \hat{N}\, ds$

DEFINITION: The flux of a field $\vec{F}$ across a surface S is given by: $\iint_S \vec{F} \cdot \hat{N}\, dS$

NOTE: Like circulation or work, flux depends on orientation of surface (direction of $\hat{N}$.)

THEOREM: Friendly formulas for flux:

If S is determined by $z = f(x, y)$, then:

- $\iint_S \vec{F} \cdot \hat{N}\, dS = \iint_R \vec{F}(x, y, f(x, y)) \cdot \langle -f_x(x, y), -f_y(x, y), 1\rangle\, dA$ (*upward* normal)

- $\iint_S \vec{F} \cdot \hat{N}\, dS = \iint_R \vec{F}(x, y, f(x, y)) \cdot \langle f_x(x, y), f_y(x, y), -1\rangle\, dA$ (*downward* normal)

If S is given by $\vec{r}(u, v)$, then:

- $\iint_S \vec{F} \cdot \hat{N}\, dS = \pm \iint_R \vec{F}(x(u, v), y(u, v), z(u, v)) \cdot (\vec{r}_u \times r_v)(u, v)\, dA$, the $\pm$ depends on orientation.

EXAMPLE 14: Let S be the portion of $x + 2y + 3z = 12$ which lies in the first octant.

Set-up, but do not evaluate, a double integral to find the **upward** flux of $\vec{F}(x, y, z) = \left\langle -2xz, xz, y^2 \right\rangle$ across S.

Ans: $\displaystyle\int_0^6 \int_0^{-2y+12} \left\langle -2x\left(-\frac{1}{3}x - \frac{2}{3}y + 4\right), x\left(-\frac{1}{3}x - \frac{2}{3}y + 4\right), y^2 \right\rangle \cdot \left\langle \frac{1}{3}, \frac{2}{3}, 1 \right\rangle dx\, dy = \int_0^6 \int_0^{-2y+12} y^2\, dx\, dy$

EXAMPLE 15: Let S be the boundary of the solid bounded by $z = 4 - x^2 - y^2$ and $z = 3$.

Set-up, but do not evaluate, a sum of integrals to find the **outward** flux of $\vec{F}(x, y.z) = \langle 3x, 2y, z\rangle$ across S.

HINT: Use polar coordinates.

Ans: $\displaystyle\int_0^{2\pi}\int_0^1 \left(2r^2\cos^2(\theta) + 3r^2 + 4\right)\, r\, dr\, d\theta + \int_0^{2\pi}\int_0^1 (-3) r\, dr\, d\theta$

EXAMPLE 16: Let S be given by $\vec{r}(u, v) = \langle 4\cos(u), 4\sin(u), v\rangle$ for $-\pi \le u < \pi$ and $0 \le v \le 5$.

Find the **outward** flux of $\vec{F}(x, y, z) = \langle z, y, z\rangle$ across S.

Ans: $\displaystyle\int_{-\pi}^{\pi}\int_0^5 \langle v, 4\sin(u), v\rangle \cdot \langle 4\cos(u), 4\sin(u), 0\rangle\, dv\, du = \int_{-\pi}^{\pi}\int_0^5 \left(4v\cos(u) + 16\sin^2(u)\right)\, dv\, du = 80\pi$

HOMEWORK: Section 17.6: 9 - 73 every other odd; 74*, 75*